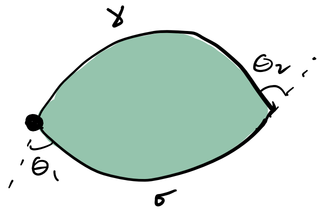
Applications:

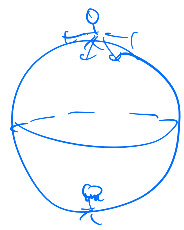
$\ulcorner$ ① $\chi(S) > 0$, S cpct or $\Rightarrow S \cong S^2$

② $\iint_S K d\sigma = 2\pi \chi(S)$ $\lrcorner$

1. S cpct or. $K \geq 0$ not $\equiv 0$ $\Rightarrow$ $S \underset{\text{homeo}}{\cong} S^2$

2. If $K \leq 0$, S can have no geodesic lunes





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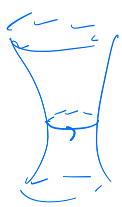
$$\cancel{\int k_g ds}^{\,0} \quad \theta_1 + \theta_2 = 2\pi - \iint K d\sigma$$

$$\Rightarrow \theta_1 + \theta_2 \geq 2\pi$$

$$\Rightarrow \theta_1 = \theta_2 = \pi \quad \text{b/c} \quad \theta_i \leq \pi$$

$\Rightarrow \sigma = \gamma$ by uniqueness of geodesics $\rightarrow \leftarrow$ $\lrcorner$



3. If S homeo $S^1 \times \mathbb{R}$, $K < 0$, then S has at most one simple closed geodesic.

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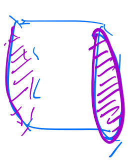
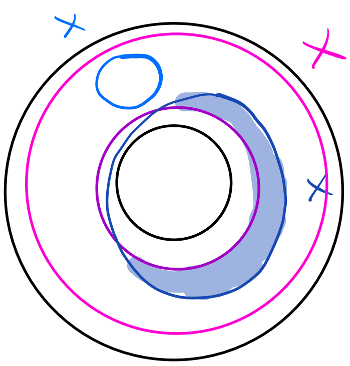


χ(blue region) $= 2$

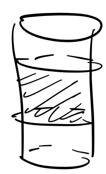
χ(blue) $= 0$



$\bigcirc > 2\pi \rightarrow \leftarrow$

$\bigcirc > \bigcirc \rightarrow \leftarrow$

no lunes $\rightarrow \leftarrow$



extension: In fact, we see if $K \leq 0$, and S has two simple closed geodesics, then the region in between has $K = 0$. $\lrcorner$

4. If $K>0$, S compact then any two simple closed geodesics intersect.

$\top$ Else, if R is the surface between them

$$0 = \chi(R) - \iint_R K$$

$$\Rightarrow \chi(R) > 0$$

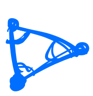
but $\chi(R) \leq 0$ b/c if you glue on two disks to ∂R, you get a closed surface $\tilde{R}$ so $\chi(\tilde{R}) \leq 2$, but $\chi(\tilde{R}) = \chi(R) + 2$. $\lrcorner$

5. Interior angle sums in geodesic Δ are

- Equal to π if $K=0$
- Greater than π if $K>0$
- less than π if $K<0$

$\top$

$$\Sigma \theta_i = 2\pi - \iint_\Delta K\,d\sigma \qquad \varphi_i = \pi - \theta_i \text{ are interior angles}$$



$$3\pi - \Sigma \varphi_i = 2\pi - \iint_\Delta K\,d\sigma$$

$$\iint_\Delta K + \pi = \Sigma \varphi_i \quad \lrcorner$$

2-D spaceforms & the derivative of the shape op in normal coords.

$L_\partial g = 2gB$

$B = \frac{1}{2} g^{-1} L_\partial g$

$$(L_r B)(X) = L_\partial(B(X)) - B(L_\partial X)$$

$$= D_r D_x r - \cancel{D_{D_x r} r} - D_{D_r x} r + \cancel{D_{D_x r} r}$$

$$= R(r, X) r + D_x D_r r - D_{D_x r} r$$

$$= -R_r X - B^2 X$$ <u>why?</u> b/c $R_\partial = K\,\pi_\partial^\perp$ for surfaces.

<u>eg</u> $g = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\lambda_r}{\lambda} \end{bmatrix}$

$$B_r + B^2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\lambda_{rr}}{\lambda} \end{bmatrix}$$

$$(B_r + B^2)(\partial_\theta) = \left(\frac{\lambda_{rr}}{\lambda}\right)(\partial_\theta)$$

$$K = -\frac{\lambda_{rr}}{\lambda}$$

⟶ solve hyperbolic plane, sphere: <u>lem</u> 2-dim'l spaceforms are unique.

<u>Recall</u> $\Omega^1_2(e_1, e_2) = K \;\langle [D_1, D_2] e_1, e_2 \rangle \;\Omega_{12}{}^{1}{}_{2}$